

Axions and “Light Shining Through a Wall”: A Detailed Theoretical Analysis

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(Dated: March 5, 2008)

We give a detailed study of axion-photon and photon-axion conversion amplitudes, which enter the analysis of “light shining through a wall” experiments. Several different calculational methods are employed and compared, and in all cases we retain a nonzero axion mass. To leading order, we find that when the photon frequency ω is very close to the axion mass m , there is a threshold cusp which significantly enhances the photon to axion conversion amplitude, by a factor $\omega/\sqrt{\omega^2 - m^2}$ relative to the corresponding axion to photon conversion process. When $m = 0$, the enhancement factor reduces to unity and the results of previous calculations are recovered. Our calculations include an exact wave matching analysis, which shows how unitarity is maintained near threshold at $\omega = m$, and a discussion of the case when the magnetic field extends into the “wall” region.

I. INTRODUCTION

The axion is a remarkable idea proposed at the end of the seventies in order to solve the strong CP problem [1]. The possible existence of the axion could help solve long-standing cosmological problems and, therefore, searching for it is an important issue [2].

Following on the seminal papers of Sikivie [3] which showed that axions can be detected through axion-photon conversion in a magnetic field, twenty years ago Van Bibber et al. proposed a “light shining through a wall” (LSW) experiment where photon regeneration could be used as an indication of an axion-photon coupling [4]. In the Van Bibber et al. setup (see Figure 1), \mathbf{B} is an external magnetic field and the wall is opaque for photons while it is transparent for the very weakly interacting axions. An x-ray version of this experiment has recently been proposed by Rabadán et al. [5], and two recent optical laser LSW experiments have reported preliminary results [6], [7].

Motivated by the current interest in LSW experiments and axion detection, we give here a detailed examination of the theory of photon-axion and axion-photon conversion processes. Our aims are in part to compare different calculational methods, and in part to examine threshold effects that appear when the axion mass is not set equal to zero.

In the LSW experiments one can distinguish seven zones, as sketched the figure. We are mainly interested in the transition zones I-II-III and V-VI-VII, namely, vacuum-magnetic field-vacuum regions where an incident photon gives rise to an exiting axion, and an incident axion gives rise to a regenerated photon. Although we will mainly focus on the theory connected with these regions, we will also briefly discuss what happens when the magnetic field enters the absorbing wall region.

The paper is organized as follows. In Sec. II we give the Lagrangian, equations of motion, and some elementary properties following from the equations of motion, such as the continuity conditions, free space kinematics, conserved probability current, and unitarity relation. In Sec. III we derive the leading order photon-axion and axion-photon conversion amplitudes by a Green function method, and in Sec. IV we repeat this derivation by a WKB/eikonal method, in both cases allowing the magnetic field to have a general dependence on z , the coordinate along the axis of the experiment. We find that when the axion mass is taken into account, the ratio of the photon-axion conversion amplitude to the axion-photon conversion amplitude is ω/k , with ω and $k = \sqrt{\omega^2 - m^2}$ respectively the photon and axion wave numbers. Since the leading order axion-photon amplitude violates unitarity near threshold at $\omega = m$, in

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Sec. V we do an all orders calculation for the case of a piecewise constant magnetic field. This gives the shape of the photon-axion conversion amplitude near threshold, and allows us to check that the unitarity constraint is obeyed. In Sec. VI we give further results following from the calculation of Sec. V. We show that the ratio of photon-axion to axion-photon amplitudes derived to lowest order is in fact exact at all orders for a piecewise constant magnetic field, we show that the all orders calculation restores unitarity near threshold, and we give some rough estimates of “light through a wall” and photon-axion conversion probabilities integrated over the threshold cusp region. In Sec. VII we briefly discuss what happens when the magnetic field penetrates the wall region; although the wall strongly absorbs photons, the weak coupling of photons to axions leads to a greatly suppressed absorption of the axion wave. Finally, in Sec. VIII we give a brief summary and suggestions for follow-up work.

II. LAGRANGIAN, EQUATIONS OF MOTION, AND UNITARITY RELATIONS

In this paper we will consider a system described by the action

$$\mathcal{S} = \int d^4x \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\phi(\partial^2 + m^2)\phi + \frac{g}{4}\phi F^{\mu\nu}\tilde{F}_{\mu\nu} \right] , \quad (1)$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, where ϕ is a real scalar field representing the axion, $F_{\mu\nu}$ is the electromagnetic field tensor, and g is the coupling constant.

The physical situation that we are interested in consists of propagating photons and axions in the presence of a static magnetic field as a background. We will suppose that this magnetic background B is pointing in the x direction, so that the relevant part of the interaction term is

$$\int d^4x \beta \phi E_x , \quad (2)$$

where $\beta = gB$ and E_x is the component of the photon electric field parallel to the background magnetic field.

Neglecting components of the photon field that do not couple to the axion through the magnetic field, and using Coulomb gauge, (1) becomes

$$\mathcal{S} = -\frac{1}{2} \int d^4x [a\partial^2 a + \phi(\partial^2 + m^2)\phi - 2\beta\phi\partial_t a] , \quad (3)$$

where a is a real potential which determines the relevant electric field component through $E_x = \partial_t a$. From this action we get the equations of motion,

$$\begin{aligned} (\partial^2 + m^2)\phi &= \beta\partial_t a , \\ \partial^2 a &= -\beta\partial_t \phi , \end{aligned} \quad (4)$$

so that, specializing to the case in which the propagation direction is the z axis, we have

$$\begin{aligned} (\partial_t^2 - \partial_z^2 + m^2)\phi(z, t) &= \beta(z)\partial_t a(z, t) , \\ (\partial_t^2 - \partial_z^2)a(z, t) &= -\beta(z)\partial_t \phi(z, t) . \end{aligned} \quad (5)$$

The entire analysis of this paper will follow from the equations of motion (5). In particular, since these are second order in z , the functions $\phi(z, t)$ and $a(z, t)$, and their first spatial derivatives, must be continuous. Since no complex numbers enter the equations, we can follow the usual method of generalizing the wave amplitudes from real to complex. Multiplying the equation for ϕ by ϕ^* , multiplying the equation for ϕ^* by ϕ and subtracting; similarly multiplying the equation for a by a^* , multiplying the equation for a^* by a and subtracting; and finally adding the two resultant equations, one finds a current conservation equation

$$\begin{aligned} \partial_t j_0 + \partial_z j_z &= 0 , \\ j_0 &= \phi^* \partial_t \phi - \partial_t \phi^* \phi + a^* \partial_t a - \partial_t a^* a + \beta(z)(a^* \phi - \phi^* a) , \\ j_z &= \partial_z \phi^* \phi - \phi^* \partial_z \phi + \partial_z a^* a - a^* \partial_z a . \end{aligned} \quad (6)$$

Specializing now to the case of an incident wave with time dependence $f(z)e^{-i\omega t}$, where f denotes the fields ϕ or A , j_0 becomes independent of time, and so $\partial_t j_0 = 0$. The current conservation equation then becomes $\partial_z j_z = 0$, which

we use to give a useful unitarity relation, as follows. In free space regions where the magnetic field vanishes (zones I, III, V, and VII of Figure 1), the z dependence of A is $e^{\pm i\omega z}$ and of ϕ is $e^{\pm ikz}$, where

$$k = \sqrt{\omega^2 - m^2} \quad . \quad (7)$$

Assuming no incident waves from the right of the region where the B field is nonvanishing, the fields to the left will be given by

$$\begin{aligned} \phi(z, t) &= e^{-i\omega t} (\phi_I e^{ikz} + \phi_R e^{-ikz}) \quad , \\ a(z, t) &= e^{-i\omega t} (a_I e^{i\omega z} + a_R e^{-i\omega z}) \quad , \end{aligned} \quad (8)$$

and the fields to the right of the B field region will be given by

$$\begin{aligned} \phi(z, t) &= e^{-i\omega t} \phi_T e^{ikz} \quad , \\ a(z, t) &= e^{-i\omega t} a_T e^{i\omega z} \quad . \end{aligned} \quad (9)$$

Substituting (8) and (9) into (6) to give j_z on the left and right, respectively, and equating these, we get the unitarity relation between incident, reflected, and transmitted wave amplitudes,

$$k|\phi_I|^2 + \omega|a_I|^2 = k(|\phi_R|^2 + |\phi_T|^2) + \omega(|a_R|^2 + |a_T|^2) \quad . \quad (10)$$

This equation will play a role in our subsequent discussion.

III. GREEN FUNCTION CALCULATION OF PHOTON-AXION AND AXION-PHOTON CONVERSION

We turn now to a calculation of the photon-axion and axion-photon conversion amplitudes, to first order in the magnetic field strength parameter $\beta(z)$.

By performing a perturbation expansion in powers of β we write the fields as follows

$$\phi(z, t) = \phi^{(0)}(z, t) + \phi^{(1)}(z, t) + \dots \quad , \quad (11a)$$

$$a(z, t) = a^{(0)}(z, t) + a^{(1)}(z, t) + \dots \quad , \quad (11b)$$

where

$$(\partial_t^2 - \partial_z^2 + m^2) \phi^{(0)}(z, t) = 0 \quad , \quad (12a)$$

$$(\partial_t^2 - \partial_z^2) a^{(0)}(z, t) = 0 \quad , \quad (12b)$$

$$(\partial_t^2 - \partial_z^2 + m^2) \phi^{(1)}(z, t) - \beta(z) \partial_t a^{(0)}(z, t) = 0 \quad , \quad (12c)$$

$$(\partial_t^2 - \partial_z^2) a^{(1)}(z, t) + \beta(z) \partial_t \phi^{(0)}(z, t) = 0 \quad . \quad (12d)$$

In the last two equations, unperturbed fields [solutions of (12a) and (12b)] are sources for perturbed ones, so we will calculate the Green function for the differential operator in (12a). Clearly, for case of the differential operator in (12b) it is enough to put $m = 0$. Thus, we are interested in the solutions $G_m(z, t)$ of

$$(\partial_t^2 - \partial_z^2 + m^2) G_m(z, t) = \delta(z) \delta(t) \quad . \quad (13)$$

A direct calculation in Fourier space shows that the desired Green function is

$$G_m(z, t) = -\frac{1}{(2\pi)^2} \int dk d\omega \frac{e^{i(kz - \omega t)}}{(\omega - \omega_k)(\omega + \omega_k)} \quad , \quad (14)$$

with $\omega_k = \sqrt{k^2 + m^2}$.

Since we want the retarded solution, that is $G_m(z, t) = 0$ for $t < 0$, we integrate on the real ω axis, with poles $\mp \omega_k$ shifted to $\mp \omega_k - i\epsilon$. When $t < 0$, closing the contour up gives 0; when $t > 0$, closing the contour down gives the retarded Green function

$$G_m^R(z, t) = -\theta(t) \left(\frac{i}{4\pi} \right) \int \frac{dk}{\omega_k} \left(e^{i(kz + \omega_k t)} - e^{i(kz - \omega_k t)} \right) \equiv -\theta(t) \Delta_m(z, t) \quad . \quad (15)$$

Writing Δ_m in a more compact form we obtain

$$\Delta_m(z, t) = \int \frac{dk}{2\pi} \frac{\sin(k|z| - \omega_k t)}{\omega_k} \quad , \quad (16)$$

and for the photon, taking $m = 0$, we find

$$G_0^R(z, t) = -\theta(t)\Delta_0(z, t) = \frac{1}{2}\theta(t - |z|) \quad . \quad (17)$$

Returning now to the solution of (12c), for the case $\phi^{(0)} = 0$ of no incident axion, we take the incident photon field as a monochromatic wave $a^{(0)} = e^{i\bar{\omega}(z-t)}$, so that the inhomogeneous equation (12c) reads

$$(\partial_t^2 - \partial_z^2 + m^2)\phi^{(1)}(z, t) = -i\bar{\omega}\beta(z)e^{i\bar{\omega}(z-t)} \quad . \quad (18)$$

The solution to (18), constructed using the Green function (15) is

$$\begin{aligned} \phi^{(1)}(z, t) &= -i\bar{\omega} \int_0^L dz' \beta(z') e^{i\bar{\omega}z'} \int_{-\infty}^{\infty} dt' G_m^R(z - z', t - t') e^{-i\bar{\omega}t'} \\ &= -i\bar{\omega} e^{-i\bar{\omega}t} \int_0^L dz' \beta(z') e^{i\bar{\omega}z'} \int_{-\infty}^{\infty} \frac{dk}{\omega_k} \left(\frac{-i}{4\pi} \right) e^{ik(z-z')} \\ &\quad \times \int_{-\infty}^{\infty} dt' \theta(t - t') e^{i\bar{\omega}(t-t')} \left(e^{i(t-t')\omega_k} - e^{-i(t-t')\omega_k} \right) \quad , \end{aligned} \quad (19)$$

where we have assumed that $\beta(z) = 0$ outside the interval $0 \leq z \leq L$.

The integral on t' can be done straightforwardly by defining $T = t - t'$ and noting that, due to the θ function, there are contributions only for $T > 0$. Then

$$\begin{aligned} \phi^{(1)}(z, t) &= -i\bar{\omega} e^{-i\bar{\omega}t} \int_0^L dz' \beta(z') e^{i\bar{\omega}z'} \int_{-\infty}^{\infty} \frac{dk}{\omega_k} \left(\frac{-i}{4\pi} \right) e^{ik(z-z')} \left(\frac{2i\omega_k}{\omega_k^2 - \bar{\omega}^2} \right) \\ &= -i\bar{\omega} e^{-i\bar{\omega}t} \int_0^L dz' \beta(z') e^{i\bar{\omega}z'} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(z-z')}}{(k - \sqrt{\bar{\omega}^2 - m^2})(k + \sqrt{\bar{\omega}^2 - m^2})} \quad . \end{aligned} \quad (20)$$

The integral on k can be performed as usual in the complex plane. The boundary condition that $a^{(0)}$ vanishes in the infinite past can be included by giving $\bar{\omega}$ a small positive imaginary part $\bar{\omega} \rightarrow \bar{\omega} + i\epsilon$, $\epsilon > 0$, so that

$$\pm \sqrt{(\bar{\omega} + i\epsilon)^2 - m^2} \sim \pm \sqrt{\bar{\omega}^2 - m^2} \pm \frac{i\epsilon\bar{\omega}}{\sqrt{\bar{\omega}^2 - m^2}} = \pm \sqrt{\bar{\omega}^2 - m^2} \pm i\epsilon' \quad , \quad \epsilon' > 0 \quad . \quad (21)$$

We see that for $z - z' > 0$ the contour can be closed up, circling the pole at $k = \sqrt{\bar{\omega}^2 - m^2} + i\epsilon'$, while for $z - z' < 0$ the contour can be closed down, circling the pole at $k = -\sqrt{\bar{\omega}^2 - m^2} - i\epsilon'$. Equation (20) thus gives

$$\begin{aligned} \phi^{(1)}(z, t) &= \frac{\bar{\omega}}{2\sqrt{\bar{\omega}^2 - m^2}} \left[e^{-i\bar{\omega}t + iz\sqrt{\bar{\omega}^2 - m^2}} \int_0^L dz' \theta(z - z') \beta(z') e^{iz'(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2})} \right. \\ &\quad \left. + e^{-i\bar{\omega}t - iz\sqrt{\bar{\omega}^2 - m^2}} \int_0^L dz' \theta(z' - z) \beta(z') e^{iz'(\bar{\omega} + \sqrt{\bar{\omega}^2 - m^2})} \right] \\ &= \frac{\bar{\omega}}{2\sqrt{\bar{\omega}^2 - m^2}} \left[e^{-i\bar{\omega}t + iz\sqrt{\bar{\omega}^2 - m^2}} \int_0^z dz' \beta(z') e^{iz'(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2})} \right. \\ &\quad \left. + e^{-i\bar{\omega}t - iz\sqrt{\bar{\omega}^2 - m^2}} \int_z^L dz' \beta(z') e^{iz'(\bar{\omega} + \sqrt{\bar{\omega}^2 - m^2})} \right] \quad . \end{aligned} \quad (22)$$

We see that the axion amplitude has a transmitted and a reflected part,

$$\phi^{(1)}(z, t) = \phi_{\text{Trans}}^{(1)}(z, t) + \phi_{\text{Reflect}}^{(1)}(z, t) \quad , \quad (23)$$

with

$$\begin{aligned} \phi_{\text{Trans}}^{(1)}(z, t) &= \frac{\bar{\omega}}{2\sqrt{\bar{\omega}^2 - m^2}} e^{-i\bar{\omega}t + iz\sqrt{\bar{\omega}^2 - m^2}} \int_0^z dz' \beta(z') e^{iz'(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2})} \quad , \\ \phi_{\text{Reflect}}^{(1)}(z, t) &= \frac{\bar{\omega}}{2\sqrt{\bar{\omega}^2 - m^2}} e^{-i\bar{\omega}t - iz\sqrt{\bar{\omega}^2 - m^2}} \int_z^L dz' \beta(z') e^{iz'(\bar{\omega} + \sqrt{\bar{\omega}^2 - m^2})} \quad , \end{aligned} \quad (24)$$

for general z , or for z values outside the magnetic field region,

$$\begin{aligned}\phi_{\text{Trans}}^{(1)}(z, t) &= \frac{\bar{\omega}}{2\sqrt{\bar{\omega}^2 - m^2}} e^{-i\bar{\omega}t + iz\sqrt{\bar{\omega}^2 - m^2}} \int_0^L dz' \beta(z') e^{iz'(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2})}, \quad z \geq L, \\ \phi_{\text{Reflect}}^{(1)}(z, t) &= \frac{\bar{\omega}}{2\sqrt{\bar{\omega}^2 - m^2}} e^{-i\bar{\omega}t - iz\sqrt{\bar{\omega}^2 - m^2}} \int_0^L dz' \beta(z') e^{iz'(\bar{\omega} + \sqrt{\bar{\omega}^2 - m^2})}, \quad z \leq 0.\end{aligned}\quad (25)$$

From (25) we see that if the exponential in the integral for $\phi_{\text{Trans}}^{(1)}$ is rapidly varying on the scale over which $\beta(z)$ varies, the axion transmission is strongly suppressed. This statement can be given a quantitative form by an integration by parts,

$$\int_0^L dz' \beta(z') e^{iz'(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2})} = i \int_0^L dz' \frac{\beta'(z')}{\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2}} e^{iz'(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2})}, \quad (26)$$

where we have used the fact that $\beta(0) = \beta(L) = 0$ to drop the surface terms. If we assume that $|\beta'(z)/\beta(z)| < 1/h$, with h a measure of the characteristic distance over which the magnetic field varies, and that $|\beta(z)| < \beta_M$, with β_M a measure of the maximum magnetic field, then substitution of (26) into (25) gives the inequality

$$|\phi_{\text{Trans}}^{(1)}| \leq \frac{\beta_M L}{2} \frac{\bar{\omega}}{\sqrt{\bar{\omega}^2 - m^2}} \frac{1}{h(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2})}. \quad (27)$$

Let us now consider $\beta(z) = \beta$, a constant. The previous integrals can be done straightforwardly, and we obtain for the transmitted part

$$\phi_{\text{Trans}}^{(1)}(z, t) = \frac{\beta \bar{\omega} e^{i(z\sqrt{\bar{\omega}^2 - m^2} - \bar{\omega}t)}}{m^2 \sqrt{\bar{\omega}^2 - m^2}} e^{\frac{iL}{2}(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2})} (\bar{\omega} + \sqrt{\bar{\omega}^2 - m^2}) \sin\left(\frac{L}{2}(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2})\right), \quad (28)$$

which has the modulus squared

$$|\phi_{\text{Trans}}^{(1)}(z, t)|^2 = \frac{\beta^2}{m^4} \left(\frac{\bar{\omega}^2}{\bar{\omega}^2 - m^2} \right) (\bar{\omega} + \sqrt{\bar{\omega}^2 - m^2})^2 \sin^2\left(\frac{L}{2}(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2})\right). \quad (29)$$

Let us recall now that the unitarity relation of (10) requires, for an incident photon amplitude of magnitude unity, that the transmitted axion amplitude obey the unitarity constraint $1 \geq (k/\omega)|\phi_T|^2$, which in the notation of this section is $1 \geq (\sqrt{\bar{\omega}^2 - m^2}/\bar{\omega})|\phi^{(1)}(z \geq L, t)|^2$. From (29) we see, however, that near threshold at $\bar{\omega} \simeq m$, we have

$$(\sqrt{\bar{\omega}^2 - m^2}/\bar{\omega})|\phi^{(1)}(z \geq L, t)|^2 = \frac{\beta^2}{m\sqrt{\bar{\omega}^2 - m^2}} \sin^2(mL/2), \quad (30)$$

which approaches infinity as $\bar{\omega} \rightarrow m$. Thus the lowest order expression for the transmitted axion amplitude violates unitarity near threshold; an all orders calculation, given below, is needed to see how unitarity is restored, and to give the shape of the axion transmission probability near threshold where $k = \sqrt{\bar{\omega}^2 - m^2} \sim \beta^2/m$.

To complete the calculation, we will give the corresponding result for the photon field amplitude induced by an axion wave incident on the magnetic field region, with no incident photon (*i.e.*, $a^{(0)} = 0$). Using the zero mass retarded Green function G_0^R of (17) with the source term $-\beta(z)\partial_t\phi^{(0)}(z, t) = i\bar{\omega}\beta(z)e^{i(\sqrt{\bar{\omega}^2 - m^2}z - \bar{\omega}t)}$ in (12d), we have

$$\begin{aligned}a^{(1)}(z, t) &= \int_0^L dz' \int_{-\infty}^{\infty} dt' G_0^R(z - z', t - t') i\bar{\omega}\beta(z') e^{i(\sqrt{\bar{\omega}^2 - m^2}z' - \bar{\omega}t')} \\ &= \int_0^L dz' \int_{-\infty}^{\infty} dt' \frac{1}{2} \theta(t - t' - |z - z'|) i\bar{\omega}\beta(z') e^{i(\sqrt{\bar{\omega}^2 - m^2}z' - \bar{\omega}t')} \\ &= \frac{i\bar{\omega} e^{-i\bar{\omega}t}}{2} \int_0^L dz' \beta(z') e^{iz'\sqrt{\bar{\omega}^2 - m^2}} \int_{|z-z'|}^{\infty} e^{i\bar{\omega}T} dT.\end{aligned}\quad (31)$$

Finally, performing the integral on $T = t - t'$, we find for the transmitted photon amplitude at $z \geq L$,

$$a_{\text{Trans}}^{(1)}(z, t) = -\frac{e^{i\bar{\omega}(z-t)}}{2} \int_0^L dz' \beta(z') e^{iz'(\sqrt{\bar{\omega}^2 - m^2} - \bar{\omega})}. \quad (32)$$

We see that when the axion mass is nonzero, the ratio of transmitted wave amplitude magnitudes is $|\phi_{\text{Trans}}^{(1)}/a_{\text{Trans}}^{(1)}| = \bar{\omega}/\sqrt{\bar{\omega}^2 - m^2} > 1$. When the axion mass m vanishes, the transmitted amplitudes (28) and (32) are identical up to a sign, as analyzed in greater detail by Guendelman [8] using his demonstration of an axion-photon duality.

IV. WKB/EIKONAL CALCULATION OF PHOTON-AXION AND AXION-PHOTON CONVERSION

In this section we calculate the photon-axion and axion-photon conversion amplitudes, using a WKB/eikonal method. As an alternative method of solving the equations of motion, with $e^{-i\omega t}$ time dependence, we consider the following Ansatz

$$\phi(z, t) = \phi_0 e^{i\theta(z) - i\omega t}, \quad a(z, t) = a_0 e^{i\chi(z) - i\omega t} . \quad (33)$$

The equations of motion (5) now read

$$\begin{aligned} (\omega^2 - m^2 - (\theta')^2 + i\theta'')\phi - i\beta(z)\omega a &= 0 , \\ (\omega^2 - (\chi')^2 + i\chi'')a + i\beta(z)\omega\phi &= 0 . \end{aligned} \quad (34)$$

Expanding $\theta(z)$ and $\chi(z)$ in powers of β ,

$$\theta(z) = \sqrt{\omega^2 - m^2}z + \theta^{(1)}(z) + \dots, \quad \chi(z) = \omega z + \chi^{(1)}(z) + \dots, \quad (35)$$

where the index (1) indicates the first order term in the expansion in β , equations (34) become

$$\begin{pmatrix} 2\sqrt{\omega^2 - m^2}(\theta^{(1)})' e^{iz\sqrt{\omega^2 - m^2}} & i\beta(z)\omega e^{iz\omega} \\ -i\beta(z)\omega e^{iz\sqrt{\omega^2 - m^2}} & 2\omega(\chi^{(1)})' e^{iz\omega} \end{pmatrix} \begin{pmatrix} \phi_0 \\ a_0 \end{pmatrix} = 0 , \quad (36)$$

where for the moment we have dropped second derivatives of phases (as indicated below by a subscript WKB), and only first order contributions in β are considered.

Non-trivial solutions satisfy the null determinant condition

$$(\theta^{(1)})'(\chi^{(1)})' = \frac{\omega\beta^2(z)}{4\sqrt{\omega^2 - m^2}} . \quad (37)$$

From the second row of (36) we get

$$\chi^{(1)}(z) = \frac{i\phi_0}{2a_0} \int^z dz' \beta(z') e^{iz'(\sqrt{\omega^2 - m^2} - \omega)} , \quad (38)$$

while from the first row of (36) (or using the null determinant condition) we get

$$\theta^{(1)}(z) = \frac{-ia_0}{2\phi_0} \frac{\omega}{\sqrt{\omega^2 - m^2}} \int^z dz' \beta(z') e^{iz'(\omega - \sqrt{\omega^2 - m^2})} . \quad (39)$$

This integral is the same as the one appearing in the transmitted axion wave in (24). In fact, we can now explicitly write the axion field to first order in β , in the approximation of neglecting second derivatives in (36), as

$$\begin{aligned} \phi_{\text{WKB}}(z, t) &\simeq \phi_0 e^{i(z\sqrt{\omega^2 - m^2} - \omega t)} (1 + i\theta^{(1)}(z)) \\ &\simeq \phi_0 e^{i(z\sqrt{\omega^2 - m^2} - \omega t)} \left(1 + \frac{a_0}{2\phi_0} \frac{\omega}{\sqrt{\omega^2 - m^2}} \int^z dz' \beta(z') e^{iz'(\omega - \sqrt{\omega^2 - m^2})} \right) , \end{aligned} \quad (40)$$

or an equivalent expression,

$$\phi_{\text{WKB}}(z, t) - \phi^{(0)}(z, t) \simeq \frac{a_0}{2} \frac{\omega}{\sqrt{\omega^2 - m^2}} e^{i(z\sqrt{\omega^2 - m^2} - \omega t)} \int^z dz' \beta(z') e^{iz'(\omega - \sqrt{\omega^2 - m^2})} . \quad (41)$$

The photon field satisfies a similar relation, namely

$$a_{\text{WKB}}(z, t) - a^{(0)}(z, t) \simeq -\frac{\phi_0}{2} e^{i\omega(z-t)} \int^z dz' \beta(z') e^{iz'(\sqrt{\omega^2 - m^2} - \omega)} . \quad (42)$$

With the substitution $\omega \rightarrow \bar{\omega}$, and taking the lower limit of integration as $z = 0$ for the case when $\beta(z)$ vanishes for $z \leq 0$, these agree with the transmission amplitudes obtained by the Green function method.

Finally, let us perform the calculation, again up to first order in β , but preserving the second derivatives of phases, so that the exact first order amplitude is obtained, including the reflection amplitudes calculated by the Green function method. From (34) we obtain

$$\phi_0 (\theta^{(1)})'' + 2i\phi_0 \sqrt{\omega^2 - m^2} (\theta^{(1)})' = a_0 \omega \beta(z) e^{i(\omega - \sqrt{\omega^2 - m^2})z} , \quad (43)$$

which implies

$$\left(\phi_0 (\theta^{(1)})' e^{2iz\sqrt{\omega^2 - m^2}} \right)' = a_0 \omega \beta(z) e^{i(\omega + \sqrt{\omega^2 - m^2})z} , \quad (44)$$

with the integral

$$i\phi_0 (\theta^{(1)})' = iC e^{-2iz\sqrt{\omega^2 - m^2}} + ia_0 \omega e^{-2iz\sqrt{\omega^2 - m^2}} \int_0^z dz' \beta(z') e^{i(\omega + \sqrt{\omega^2 - m^2})z'} . \quad (45)$$

Integrating again, the phase $\theta^{(1)}$ is given by

$$\begin{aligned} i\phi_0 \theta^{(1)} &= \tilde{C} + \frac{iC}{\sqrt{\omega^2 - m^2}} e^{-iz\sqrt{\omega^2 - m^2}} \sin \left(z\sqrt{\omega^2 - m^2} \right) \\ &\quad + ia_0 \omega \int_0^z dz' e^{-2iz'\sqrt{\omega^2 - m^2}} \int_0^{z'} dz'' \beta(z'') e^{i(\omega + \sqrt{\omega^2 - m^2})z''} . \end{aligned} \quad (46)$$

Rewriting the final term by an integration by parts, we get

$$\begin{aligned} i\phi_0 \theta^{(1)} &= \tilde{C} + \frac{iC}{\sqrt{\omega^2 - m^2}} e^{-iz\sqrt{\omega^2 - m^2}} \sin \left(z\sqrt{\omega^2 - m^2} \right) \\ &\quad - \frac{a_0 \omega}{2\sqrt{\omega^2 - m^2}} e^{-2iz\sqrt{\omega^2 - m^2}} \int_0^z dz' \beta(z') e^{iz'(\omega + \sqrt{\omega^2 - m^2})} \\ &\quad + \frac{a_0 \omega}{2\sqrt{\omega^2 - m^2}} \int_0^z dz' \beta(z') e^{iz'(\omega - \sqrt{\omega^2 - m^2})} . \end{aligned} \quad (47)$$

Note that the last integral in (47) is what we found in (39), where second derivatives were not considered.

The integration constants can now be determined by the boundary conditions, and in order to do this, let us write the complete solution for the axion amplitude,

$$\phi(z, t) = e^{-i\omega t + iz\sqrt{\omega^2 - m^2}} \left(\phi_0 + i\phi_0 \theta^{(1)} + \dots \right) . \quad (48)$$

Substituting (47) and writing explicitly the left and right movers, we have

$$\begin{aligned} \phi(z, t) &= e^{i(z\sqrt{\omega^2 - m^2} - \omega t)} \left[\phi_0 + \tilde{C} + \frac{C}{2\sqrt{\omega^2 - m^2}} + \frac{a_0 \omega}{2\sqrt{\omega^2 - m^2}} \int_0^z dz' \beta(z') e^{iz'(\omega - \sqrt{\omega^2 - m^2})} \right] \\ &\quad - e^{-i(z\sqrt{\omega^2 - m^2} + \omega t)} \left[\frac{C}{2\sqrt{\omega^2 - m^2}} + \frac{a_0 \omega}{2\sqrt{\omega^2 - m^2}} \int_0^z dz' \beta(z') e^{iz'(\omega + \sqrt{\omega^2 - m^2})} \right] . \end{aligned} \quad (49)$$

Since we do not have incident (or right moving) axions at $z = 0$, the following condition must be satisfied

$$\phi_0 + \tilde{C} + \frac{C}{2\sqrt{\omega^2 - m^2}} = 0 , \quad (50)$$

while for $z > L$, there must be only right moving axions, and therefore

$$C + a_0 \omega \int_0^L dz' \beta(z') e^{iz'(\omega + \sqrt{\omega^2 - m^2})} = 0 . \quad (51)$$

Solving (50) and (51) for the integration constants C , \tilde{C} , and substituting back into (47), we obtain finally

$$\begin{aligned} \phi(z, t) &= e^{i(z\sqrt{\omega^2 - m^2} - \omega t)} \frac{a_0 \omega}{2\sqrt{\omega^2 - m^2}} \int_0^z dz' \beta(z') e^{iz'(\omega - \sqrt{\omega^2 - m^2})} \\ &\quad + e^{-i(z\sqrt{\omega^2 - m^2} + \omega t)} \frac{a_0 \omega}{2\sqrt{\omega^2 - m^2}} \int_z^L dz' \beta(z') e^{iz'(\omega + \sqrt{\omega^2 - m^2})} , \end{aligned} \quad (52)$$

in agreement with the transmitted and reflected waves obtained in (24) by the Green function method.

When the axion mass is very small, so that $m \ll \omega$, and $\beta(z)$ is smooth, the reflected axion wave amplitude is much smaller than the transmitted axion wave amplitude, because of the rapid oscillation of the phase factor $e^{iz'(\omega + \sqrt{\omega^2 - m^2})}$. This small amplitude is what is neglected in making the WKB approximation of neglecting second derivatives of the phases θ , χ in going from (34) to (36).

V. ALL ORDERS CALCULATION: DETAILS

We turn in this section to an all orders calculation of photon-axion and axion-photon conversion, for the case of a magnetic field that vanishes for $z < 0$ and $z > L$, and is a constant B , so that $gB = \beta$ is constant, in the interval $0 < z < L$. We assume a time dependence $e^{-i\omega t}$ throughout, so that the equations of motion have solutions of the form $f(z)e^{-i\omega t}$, where f denotes the fields ϕ or a ; henceforth in this section, the time-dependence factor $e^{-i\omega t}$ will not be shown explicitly.

Referring back to (8), the photon and axion waves in the region $z \leq 0$ have the form

$$\begin{aligned}\phi(z) &= \phi_I e^{ikz} + \phi_R e^{-ikz} \quad , \\ a(z) &= a_I e^{i\omega z} + a_R e^{-i\omega z} \quad .\end{aligned}\tag{53}$$

Generalizing (9) to allow incoming photon and axion waves from the right (which will be equated to zero later in the calculation), the photon and axion waves in the region $z \geq L$ have the form

$$\begin{aligned}\phi(z) &= \phi_T e^{ikz} + \tilde{\phi} e^{-ikz} \quad , \\ a(z) &= a_T e^{i\omega z} + \tilde{a} e^{-i\omega z} \quad .\end{aligned}\tag{54}$$

In the region $0 \leq z \leq L$, the propagation eigenmodes are obtained by solving the coupled equations (5), which with the assumed $e^{-i\omega t}$ time dependence take the form

$$\begin{aligned}(-\omega^2 - \partial_z^2 + m^2)\phi(z) &= -i\omega\beta a(z) \quad , \\ (-\omega^2 - \partial_z^2)a(z) &= i\omega\beta\phi(z) \quad .\end{aligned}\tag{55}$$

Assuming an e^{iKz} dependence of both $\phi(z)$ and $a(z)$, by substituting $\phi(z) = \phi e^{iKz}$, $a(z) = a e^{iKz}$, we find the coupled eigenmode equations

$$\begin{aligned}(-\omega^2 + K^2 + m^2)\phi &= -i\omega\beta a, \\ (-\omega^2 + K^2)a &= i\omega\beta\phi \quad ,\end{aligned}\tag{56}$$

which require K to obey the quartic equation

$$(-\omega^2 + K^2 + m^2)(-\omega^2 + K^2) = \omega^2\beta^2 \quad .\tag{57}$$

Solving this, we find that the fields propagate with four possible values for K , either $K = \pm k_+$ or $K = \pm k_-$, with

$$(k_+)^2 = \omega^2 - \frac{m^2}{2} \left(1 - \sqrt{1 + x^2}\right) \quad , \quad (k_-)^2 = \omega^2 - \frac{m^2}{2} \left(1 + \sqrt{1 + x^2}\right) \quad ,\tag{58}$$

where

$$x = 2\beta\omega/m^2 \quad .\tag{59}$$

For zero magnetic field, we have $x = 0$ and then k_+ gives the photon mode, and k_- gives the axion mode. The most general solution in the magnetic field region is

$$\phi(z) = \Phi_0^+ e^{izk_+} + \Phi_0^- e^{izk_-} + \varphi_0^+ e^{-izk_+} + \varphi_0^- e^{-izk_-} \quad ,\tag{60}$$

$$a(z) = A_0^+ e^{izk_+} + A_0^- e^{izk_-} + a_0^+ e^{-izk_+} + a_0^- e^{-izk_-} \quad .\tag{61}$$

There are restrictions on the integration constants coming from the equations of motion (56); they are

$$\Phi_0^+ = \delta A_0^+, \quad \varphi_0^+ = \delta a_0^+, \quad A_0^- = \delta \Phi_0^-, \quad a_0^- = \delta \varphi_0^- \quad ,\tag{62}$$

with

$$\delta = \frac{-ix}{1 + \sqrt{1+x^2}} \quad . \quad (63)$$

Thus, the solutions in the magnetic field region take the form

$$\phi(z) = \delta A_0^+ e^{izk_+} + \Phi_0^- e^{izk_-} + \delta a_0^+ e^{-izk_+} + \varphi_0^- e^{-izk_-} \quad , \quad (64)$$

$$a(z) = A_0^+ e^{izk_+} + \delta \Phi_0^- e^{izk_-} + a_0^+ e^{-izk_+} + \delta \varphi_0^- e^{-izk_-} \quad . \quad (65)$$

For $x \rightarrow 0$ we see that $\delta \rightarrow 0$, so that $k_+ \rightarrow \omega$ and $k_- \rightarrow k$, and we recover free photons and axions. An important property of the above equations is that when the sign of the square root $\sqrt{1+x^2}$ is changed from $+$ to $-$, the modes k_+ and k_- are interchanged, that is, $k_+ \leftrightarrow k_-$, and also δ is replaced by $\frac{-ix}{1-\sqrt{1+x^2}} = \frac{1+\sqrt{1+x^2}}{-ix} = 1/\delta$, that is $\delta \leftrightarrow 1/\delta$. This property will be used in Appendix B to show that the S -matrix elements governing all physical effects are independent of the branch choice of the square root (a behavior similar to that found in the study of vacuum birefringence effects [9, 10, 11] when one does a wave-matching calculation [12, 13] for the case of a rotating magnetic field).

We now impose continuity conditions on fields and spatial derivatives at $z = 0$, where β changes from zero to nonzero, and at $z = L$, where β changes back to zero.

A. Continuity conditions at $z = 0$

At $z = 0$, the continuity conditions for the fields are

$$\phi_I + \phi_R = \delta A_0^+ + \Phi_0^- + \delta a_0^+ + \varphi_0^- \quad , \quad (66)$$

$$a_I + a_R = A_0^+ + \delta \Phi_0^- + a_0^+ + \delta \varphi_0^- \quad , \quad (67)$$

while the continuity conditions for first spatial derivatives are

$$k(\phi_I - \phi_R) = \delta k_+(A_0^+ - a_0^+) + k_-(\Phi_0^- - \varphi_0^-) \quad , \quad (68)$$

$$\omega(a_I - a_R) = k_+(A_0^+ - a_0^+) + \delta k_-(\Phi_0^- - \varphi_0^-) \quad . \quad (69)$$

This set of four equations can be recast in matrix form as

$$M_1 \begin{pmatrix} \phi_I \\ \phi_R \\ a_I \\ a_R \end{pmatrix} = M_2 \begin{pmatrix} \Phi_0^- \\ \varphi_0^- \\ A_0^+ \\ a_0^+ \end{pmatrix} \quad , \quad (70)$$

where the matrices M_1 and M_2 can be read off from (66-69), and are given in Appendix A. It is straightforward to invert the matrix M_1 , and defining $M_{12} = M_1^{-1}M_2$, (70) can be rewritten as

$$\begin{pmatrix} \phi_I \\ \phi_R \\ a_I \\ a_R \end{pmatrix} = M_{12} \begin{pmatrix} \Phi_0^- \\ \varphi_0^- \\ A_0^+ \\ a_0^+ \end{pmatrix} \quad , \quad (71)$$

with the matrix M_{12} given in Appendix A.

B. Continuity conditions at $z = L$

At $z = L$, the continuity conditions for the fields are

$$\delta A_0^+ e^{iLk_+} + \Phi_0^- e^{iLk_-} + \delta a_0^+ e^{-iLk_+} + \varphi_0^- e^{-iLk_-} = \phi_T e^{ikL} + \tilde{\phi} e^{-ikL} \quad , \quad (72)$$

$$A_0^+ e^{iLk_+} + \delta \Phi_0^- e^{iLk_-} + a_0^+ e^{-iLk_+} + \delta \varphi_0^- e^{-iLk_-} = a_T e^{i\omega L} + \tilde{a} e^{-i\omega L} \quad , \quad (73)$$

while the continuity conditions for first spatial derivatives are

$$\delta k_+(A_0^+ e^{iLk_+} - a_0^+ e^{-iLk_+}) + k_-(\Phi_0^- e^{iLk_-} - \varphi_0^- e^{-iLk_-}) = k(\phi_T e^{ikL} - \tilde{\phi} e^{-ikL}) , \quad (74)$$

$$k_+(A_0^+ e^{iLk_+} - a_0^+ e^{-iLk_+}) + \delta k_-(\Phi_0^- e^{iLk_-} - \varphi_0^- e^{-iLk_-}) = \omega(a_T e^{i\omega L} - \tilde{a} e^{-i\omega L}) . \quad (75)$$

Again, this set of four equations can be recast in matrix form as

$$M_3 \begin{pmatrix} \Phi_0^- \\ \varphi_0^- \\ A_0^+ \\ a_0^+ \end{pmatrix} = M_4 \begin{pmatrix} \phi_T \\ \tilde{\phi} \\ a_T \\ \tilde{a} \end{pmatrix} , \quad (76)$$

where the matrices M_3 and M_4 can be read off from (72-75), and are given in Appendix A. And again, it is straight-forward to invert the matrix M_3 , and defining $M_{34} = M_3^{-1} M_4$, (76) can be rewritten as

$$\begin{pmatrix} \Phi_0^- \\ \varphi_0^- \\ A_0^+ \\ a_0^+ \end{pmatrix} = M_{34} \begin{pmatrix} \phi_T \\ \tilde{\phi} \\ a_T \\ \tilde{a} \end{pmatrix} , \quad (77)$$

with the matrix M_{34} given in Appendix A.

C. Construction of the S matrix

Combining (71) and (77), we get a matrix relation between the incoming and outgoing wave amplitudes,

$$\begin{aligned} \begin{pmatrix} \phi_I \\ \phi_R \\ a_I \\ a_R \end{pmatrix} &= M_{12} M_{34} \begin{pmatrix} \phi_T \\ \tilde{\phi} \\ a_T \\ \tilde{a} \end{pmatrix} \\ &\equiv S \begin{pmatrix} \phi_T \\ \tilde{\phi} \\ a_T \\ \tilde{a} \end{pmatrix} . \end{aligned} \quad (78)$$

From the matrices M_{12} and M_{34} given in Appendix A, explicit calculation gives the S matrix elements tabulated in Appendix B. We use them to solve the equation (78) for the following three cases.

D. Incident photon only

We assume a photon of unit amplitude incident from the left of $z = 0$, so that $a_I = 1$, but no axion incident from the left, so that $\phi_I = 0$, and no photon or axion incident from the right of $z = L$, so that $\tilde{a} = \tilde{\phi} = 0$.

With this configuration, the set of four equations that must be solved is

$$\begin{aligned} 0 &= S_{11}\phi_T + S_{13}a_T , \\ \phi_R &= S_{21}\phi_T + S_{23}a_T , \\ 1 &= S_{31}\phi_T + S_{33}a_T , \\ a_R &= S_{41}\phi_T + S_{43}a_T , \end{aligned} \quad (79)$$

from which we obtain the transmitted and reflected axion and photon amplitudes ϕ_T , ϕ_R , a_T , a_R ,

$$\begin{aligned}\phi_T &= \frac{S_{13}}{S_{13}S_{31} - S_{11}S_{33}} \ , \\ \phi_R &= \frac{S_{21}S_{13} - S_{23}S_{11}}{S_{13}S_{31} - S_{11}S_{33}} \ , \\ a_T &= \frac{-S_{11}}{S_{13}S_{31} - S_{11}S_{33}} \ , \\ a_R &= \frac{S_{41}S_{13} - S_{43}S_{11}}{S_{13}S_{31} - S_{11}S_{33}} \ .\end{aligned}\tag{80}$$

Referring to (10), these amplitudes obey the unitarity constraint

$$1 = |a_R|^2 + |a_T|^2 + \frac{k}{\omega}(|\phi_R|^2 + |\phi_T|^2) \ .\tag{81}$$

E. Incident axion only

We assume next an axion of unit amplitude incident from the left of $z = 0$, so that $\phi_I = 1$, but no photon incident from the left, so that $a_I = 0$, and again no photon or axion incident from the right of $z = L$, so that $\tilde{a} = \tilde{\phi} = 0$.

With this configuration, the set of four equations that must be solved is

$$\begin{aligned}1 &= S_{11}\phi_T + S_{13}a_T \ , \\ \phi_R &= S_{21}\phi_T + S_{23}a_T \ , \\ 0 &= S_{31}\phi_T + S_{33}a_T \ , \\ a_R &= S_{41}\phi_T + S_{43}a_T \ ,\end{aligned}\tag{82}$$

from which we obtain the transmitted and reflected axion and photon amplitudes ϕ_T , ϕ_R , a_T , a_R ,

$$\begin{aligned}\phi_T &= \frac{-S_{33}}{S_{13}S_{31} - S_{11}S_{33}} \ , \\ \phi_R &= \frac{S_{23}S_{31} - S_{21}S_{33}}{S_{13}S_{31} - S_{11}S_{33}} \ , \\ a_T &= \frac{S_{31}}{S_{13}S_{31} - S_{11}S_{33}} \ , \\ a_R &= \frac{S_{43}S_{31} - S_{41}S_{33}}{S_{13}S_{31} - S_{11}S_{33}} \ .\end{aligned}\tag{83}$$

Referring again to (10), these amplitudes obey the unitarity constraint

$$1 = |\phi_R|^2 + |\phi_T|^2 + \frac{\omega}{k}(|a_R|^2 + |a_T|^2) \ .\tag{84}$$

F. Incident photon and axion

We assume finally the more general case in which an axion of amplitude ϕ_I and a photon of amplitude a_I are both incident from the left of $z = 0$, but again no photon or axion are incident from the right of $z = L$, so that $\tilde{a} = \tilde{\phi} = 0$.

In this case, the set of four equations that must be solved is

$$\begin{aligned}\phi_I &= S_{11}\phi_T + S_{13}a_T \ , \\ \phi_R &= S_{21}\phi_T + S_{23}a_T \ , \\ a_I &= S_{31}\phi_T + S_{33}a_T \ , \\ a_R &= S_{41}\phi_T + S_{43}a_T \ ,\end{aligned}\tag{85}$$

from which we obtain the transmitted and reflected axion and photon amplitudes ϕ_T , ϕ_R , a_T , a_R ,

$$\begin{aligned}\phi_T &= \frac{-S_{33}\phi_I + S_{13}a_I}{S_{13}S_{31} - S_{11}S_{33}} \quad , \\ \phi_R &= \frac{(S_{23}S_{31} - S_{21}S_{33})\phi_I + (S_{21}S_{13} - S_{23}S_{11})a_I}{S_{13}S_{31} - S_{11}S_{33}} \quad , \\ a_T &= \frac{S_{31}\phi_I - S_{11}a_I}{S_{13}S_{31} - S_{11}S_{33}} \quad , \\ a_R &= \frac{(S_{43}S_{31} - S_{41}S_{33})\phi_I + (S_{41}S_{13} - S_{43}S_{11})a_I}{S_{13}S_{31} - S_{11}S_{33}} \quad .\end{aligned}\tag{86}$$

This is of course just a_I times the amplitudes of (80) plus ϕ_I times the amplitudes of (83), as expected from linearity. The general unitarity constraint of (10) now gives, in addition to the constraints on the S -matrix elements of (81) and (84), the additional constraint

$$\begin{aligned}0 &= k\text{Re}[(S_{23}S_{31} - S_{21}S_{33})(S_{21}S_{13} - S_{23}S_{11})^* - S_{33}S_{13}^*] \\ &\quad + \omega\text{Re}[(S_{43}S_{31} - S_{41}S_{33})(S_{41}S_{13} - S_{43}S_{11})^* - S_{11}S_{31}^*] \quad .\end{aligned}\tag{87}$$

VI. ALL ORDERS CALCULATION: DISCUSSION

We now explore various features of the all orders results obtained in the previous section. We begin by showing that to leading order in the parameter $x = 2\omega\beta/m^2$ introduced in (59), we recover the result of the Green function and WKB/eikonal calculations for the transmitted axion amplitude. When $x \ll 1$, expansion of (58) and (63) gives

$$k_- \simeq \sqrt{\omega^2 - m^2} + \mathcal{O}(x^2), \quad k_+ \simeq \omega\mathcal{O}(x^2), \quad \delta \sim -\frac{ix}{2} + \mathcal{O}(x^3) \quad ,\tag{88}$$

from which we find

$$\begin{aligned}S_{13} &= \frac{ix\omega + \sqrt{\omega^2 - m^2}}{4\sqrt{\omega^2 - m^2}}[e^{iL(\omega - \sqrt{\omega^2 - m^2})} - 1] + \mathcal{O}(x^2) \quad , \\ S_{13}S_{31} &= \mathcal{O}(x^2) \quad , \\ S_{11} &= S_{33} = 1 + \mathcal{O}(x^2) \quad .\end{aligned}\tag{89}$$

Substituting these into (80), we get for the transmitted axion amplitude when a photon of unit amplitude is incident on the magnetic field region,

$$\phi_T = \frac{ix\omega + \sqrt{\omega^2 - m^2}}{4\sqrt{\omega^2 - m^2}}(1 - e^{iL(\omega - \sqrt{\omega^2 - m^2})}) \quad ,\tag{90}$$

which can be rewritten in the usual form given in (28),

$$\phi_T = \frac{x}{2} \frac{(\omega + \sqrt{\omega^2 - m^2})}{\sqrt{\omega^2 - m^2}} e^{i\frac{L}{2}(\omega - \sqrt{\omega^2 - m^2})} \sin\left(\frac{L}{2}(\omega - \sqrt{\omega^2 - m^2})\right) \quad .\tag{91}$$

We next examine the relationship between the two amplitudes that enter into light through a wall experiments: the amplitude ϕ_T for the transmitted axion, when only a photon of unit amplitude is incident [see (80)], and the amplitude a_T for the transmitted photon, when only an axion of unit amplitude is incident [see (83)]. We have

$$\frac{\phi_T(\text{incident photon})}{a_T(\text{incident axion})} = \frac{S_{13}}{S_{31}} = -\frac{\omega}{k} e^{iL(\omega - k)} \quad ,\tag{92}$$

where we have made use of (B10). Hence the ratio of magnitudes is $\omega/\sqrt{\omega^2 - m^2}$ to all orders in x , for a piecewise constant B , generalizing the result that we found earlier to leading order.

Finally we turn to the behavior of the axion and photon transmission amplitudes when ω is very close to threshold, where we expect to find that the all orders calculation eliminates the unitarity violation that we noted at leading

order. Let us parameterize the kinematic variables of the problem in terms of a dimensionless parameter y , which takes the value -1 at threshold $\omega = m$, and 0 at the ω value where k_- vanishes. According to (58), for the product $k_+^2 k_-^2$ we have

$$k_+^2 k_-^2 = (\omega^2 - m^2/2)^2 - (m^2/2)^2(1 + x^2) = \omega^2(\omega^2 - m^2 - \beta^2) \quad , \quad (93)$$

and so k_- vanishes at $\omega^2 = m^2 + \beta^2$. (For values of ω smaller than this, down to threshold at $\omega = m$, the wave number k_- becomes imaginary.) We thus parameterize ω as follows,

$$\omega^2 = m^2 + (1 + y)\beta^2 \quad , \quad (94)$$

and assuming that $|\beta|/m \ll 1$, and that y is of order unity, we get the following approximations for the kinematic quantities of interest,

$$\omega \simeq m + \mathcal{O}(\beta^2) \quad , \quad k = \sqrt{1+y}|\beta| \quad , \quad k_+ \simeq m + \mathcal{O}(\beta^2) \quad , \quad k_- \simeq \sqrt{y}|\beta| \quad , \quad \delta \simeq -i\beta/m \quad . \quad (95)$$

The S matrix elements that are needed can now be approximated as follows, working to leading order in $|\beta|/m$ and in $\sqrt{1+y}$,

$$\begin{aligned} S_{13} &\simeq \frac{i\beta}{2|\beta|\sqrt{1+y}}(e^{iLm} - 1) \quad , \\ S_{31} &\simeq -\frac{i\beta}{2m}(1 - e^{-iLm}) \quad , \\ S_{11} &\simeq 1 - \frac{i|\beta|}{2m\sqrt{1+y}}[\sin(mL) - mL] \quad , \\ S_{33} &\simeq 1 \quad . \end{aligned} \quad (96)$$

Substituting these into (80), we find that $|\phi_T|^2$ near threshold is given by

$$|\phi_T|^2 \simeq \frac{\sin^2(\frac{1}{2}mL)}{\left(\frac{|\beta|}{m} \sin^2(\frac{1}{2}mL) + \sqrt{1+y}\right)^2 + \frac{|\beta|^2}{4m^2}(\sin(mL) - mL)^2} \quad , \quad (97)$$

or rewritten in terms of $k = |\beta|\sqrt{1+y}$,

$$|\phi_T|^2 \simeq \frac{\sin^2(\frac{1}{2}mL)}{\left(\frac{|\beta|}{m} \sin^2(\frac{1}{2}mL) + \frac{k}{|\beta|}\right)^2 + \frac{|\beta|^2}{4m^2}(\sin(mL) - mL)^2} \quad . \quad (98)$$

Note that once $y \gg 1$, or equivalently $k \gg |\beta|^2/m$, these equations reduce to

$$|\phi_T|^2 \simeq \frac{\beta^2}{\omega^2 - m^2} \sin^2(\frac{1}{2}mL) \quad , \quad (99)$$

in agreement with the leading order result (29) evaluated at $\bar{\omega} \simeq m$. Thus the leading order result is still valid when $k/m \ll 1$, but requires the correction coming from the all orders calculation when $k/m \ll |\beta|^2/m^2$.

Unitarity requires that $(k/\omega)|\phi_T|^2 \simeq (k/m)|\phi_T|^2$ should smaller from unity; from (98) we find

$$\frac{k}{m}|\phi_T|^2 \leq \frac{\frac{k}{m} \sin^2(\frac{1}{2}mL)}{\left(\frac{|\beta|}{m} \sin^2(\frac{1}{2}mL) + \frac{k}{|\beta|}\right)^2} \leq \frac{1}{4} \quad , \quad (100)$$

where we have used the fact that for positive s and t ,

$$\frac{st}{(s+t)^2} = \frac{1}{4} \frac{(s+t)^2 - (s-t)^2}{(s+t)^2} \leq \frac{1}{4} \quad . \quad (101)$$

Using the results of (92) and (98), the squared magnitude of the photon transmission amplitude, for a unit incident axion amplitude, is given near threshold by

$$|a_T|^2 \simeq \frac{\frac{k^2}{m^2} \sin^2(\frac{1}{2}mL)}{\left(\frac{|\beta|}{m} \sin^2(\frac{1}{2}mL) + \frac{k}{|\beta|}\right)^2 + \frac{|\beta|^2}{4m^2}(\sin(mL) - mL)^2} \quad . \quad (102)$$

Assuming no axion attenuation in the wall, the overall photon transmission probability $P(\omega)$ in a light through a wall experiment, very near threshold, is the product of $|\phi_T|^2$ of (98) and $|a_T|^2$ of (102). In the regime where $\beta^2/m^2 \ll k/m \ll 1$, $P(\omega)$ is well approximated by

$$P(\omega) \simeq \frac{\beta^4 \sin^4(\frac{1}{2}mL)}{2m^3(\omega - m)} \quad . \quad (103)$$

Defining $\bar{P}(\omega)$ as the average of $P(\omega)$ over an interval extending from $\omega = m$ to $\omega = m + \Delta$, integration of (103) with a lower cutoff of $k_L^2 \sim \beta^2/m^2$, or equivalently, $\omega_L - m \sim \beta^4/(2m^3)$, we find

$$\bar{P} \equiv \frac{1}{\Delta} \int_m^{m+\Delta} d\omega P(\omega) \simeq \frac{1}{\Delta} \int_{\omega_L - m}^{\Delta} d(\omega - m) P(\omega) \simeq \frac{\beta^4}{2\Delta m^3} \sin^4(\frac{1}{2}mL) \log\left(\frac{2m^3\Delta}{\beta^4}\right) \quad . \quad (104)$$

In a similar fashion, if we define the emerging axion flux from photon to axion conversion in a single B field region, for unit incident photon flux, as $F(\omega) = (k/m)|\phi_T|^2$, then in the regime where $\beta^2/m^2 \ll k/m \ll 1$, $F(\omega)$ is well approximated by

$$F(\omega) \simeq \frac{\beta^2}{mk} \sin^2(\frac{1}{2}mL) \simeq \frac{\beta^2}{m^{3/2}\sqrt{2(\omega - m)}} \sin^2(\frac{1}{2}mL) \quad . \quad (105)$$

Defining $\bar{F}(\omega)$ as the average of $F(\omega)$ over an interval extending from $\omega = m$ to $\omega = m + \Delta$, integration of (105) gives the estimate

$$\bar{F} \equiv \frac{1}{\Delta} \int_m^{m+\Delta} d\omega F(\omega) \simeq \frac{\sqrt{2}\beta^2}{\Delta^{1/2}m^{3/2}} \sin^2(\frac{1}{2}mL) \quad . \quad (106)$$

The formulas (104) and (106) will be of use for estimating the magnitude of photon-axion and axion-photon conversion effects in the threshold region.

VII. MAGNETIC FIELD PENETRATING THE WALL

Up to this point we have assumed that the magnetic field is present in a vacuum region, where there is no photon absorption. Let us now briefly examine the situation where the magnetic field and the photon absorbing “wall” overlap. In this case the photon propagation equation must include a complex dielectric constant $n = n_R + in_I$ with nonzero imaginary part n_I , and so the eigenmode equations (56) are modified to read

$$\begin{aligned} (-\omega^2 + K^2 + m^2)\phi &= -i\omega\beta a \quad , \\ (-n^2\omega^2 + K^2)a &= i\omega\beta\phi \quad , \end{aligned} \quad (107)$$

which now require K to obey the quartic equation

$$(-\omega^2 + K^2 + m^2)(-n^2\omega^2 + K^2) = \omega^2\beta^2 \quad . \quad (108)$$

For $\beta/m \ll 1$, the two types of eignmodes are a “photon-like” mode, where $-n^2\omega^2 + K^2$ is small, so that

$$-n^2\omega^2 + K^2 = \frac{\omega^2\beta^2}{-\omega^2 + K^2 + m^2} \simeq \frac{\omega^2\beta^2}{m^2 + (n^2 - 1)\omega^2} \quad , \quad (109)$$

and an “axion-like” mode, where $-\omega^2 + K^2 + m^2$ is small, so that

$$-\omega^2 + K^2 + m^2 = \frac{\omega^2\beta^2}{-n^2\omega^2 + K^2} \simeq -\frac{\omega^2\beta^2}{m^2 + (n^2 - 1)\omega^2} \quad . \quad (110)$$

Thus,

$$\begin{aligned} K_{\text{photon}} &\simeq n_R\omega + in_I\omega \quad , \\ K_{\text{axion}} &\simeq \sqrt{\omega^2 - m^2} + i\frac{n_R n_I}{\sqrt{\omega^2 - m^2}} \frac{\omega^4\beta^2}{|m^2 + (n^2 - 1)\omega^2|^2} \quad . \end{aligned} \quad (111)$$

We see that for $\omega \gg m$ and $n_I \omega > 0$, the “photon-like” mode decays as $e^{-\sigma_{\text{photon}} z}$, with $\sigma_{\text{photon}} \simeq n_I \omega$, while the “axion-like” mode decays much more weakly, as $e^{-\sigma_{\text{axion}} z}$, with

$$\sigma_{\text{axion}} \simeq \sigma_{\text{photon}} \frac{n_R \beta^2}{|n^2 - 1|^2 \omega^2} \quad . \quad (112)$$

Putting in typical numbers, $\omega \sim 1\text{eV}$, $\beta \sim 10^{-11}\text{eV}$, and $\sigma_{\text{photon}} \sim 10^5\text{cm}^{-1}$, one has $\sigma_{\text{axion}} \sim 10^{-17}\text{cm}^{-1}$, and hence there is negligible decay of the “axion-like” wave over a one meter flight path.

VIII. SUMMARY AND DISCUSSION

To summarize, we have given three different methods for calculating photon-axion and axion-photon conversion in a magnetic field. To get the lowest order transmission amplitude, the WKB method is clearly the simplest, since it involves only the integration of a first order differential equation. To also get the reflected wave in leading order, the WKB/eikonal and Green function methods are of about equal complexity. To get an all orders answer, a wave matching calculation involving 4×4 matrices is necessary. Although this calculation was done only for piecewise constant magnetic fields, by expanding the formulas of Appendix B for infinitesimal L as $S(\beta, dL) = 1 + dLG(\beta)$, and using the group property

$$S(\beta(z \leq L + dL), L + dL) = S(\beta(z \leq L), L) S(\beta(L), dL) = S(\beta(z \leq L), L) [1 + dLG(\beta(L))] \quad , \quad (113)$$

one gets a differential equation

$$\frac{dS(\beta(z \leq L), L)}{dL} = S(\beta(z \leq L), L) G(\beta(L)) \quad , \quad (114)$$

which can be integrated to give the all orders S for an arbitrary longitudinal magnetic field profile $\beta(z)$.

All three calculation methods show that when the axion mass is not set equal to zero, there is an enhancement of the photon-axion conversion amplitude near threshold at $\omega = m$. The lowest order calculations indicate a threshold cusp that violates unitarity for small enough k , and so the all orders calculation is needed to see how, for k very close to threshold, unitarity is restored. Deciding whether this threshold enhancement can be used for new experimental searches for axions will require further careful analysis. For example, when $\omega \gg m$ the light through walls probability obtained from (29) is $P(\omega \gg m) = (\beta L/2)^4$, which for large values of the effective path length mL is much larger than the probability (104) for the same geometry when ω is near threshold, assuming that the effective bandwidth Δ is of order the axion mass m . However, further investigation will be needed to see if (104) suggests ways of exploiting the threshold enhancement by using very small laser bandwidths $\Delta \ll m$. Similarly, further analysis will be needed to determine whether the enhanced axion flux near threshold given by (106) has astrophysical implications in strong magnetic field environments.

IX. ACKNOWLEDGMENTS

The authors wish to thank E. I. Guendelman for his participation in initial phases of this project, and for helpful email correspondence suggesting a WKB approach and pointing out the role of the k factor in the unitarity relation. The work of SLA was supported in part by the U. S. Department of Energy under Grant No. DE-FG02-90ER40542, JG and FM were partially supported by grants from FONDECYT 1050114 and 1060079. JL-S acknowledges the support from MEC/FULBRIGHT-FU2006-0469.

X. ADDED NOTE

Carlo Rizzo has raised the pertinent question of how the factor $\sin^2(\frac{1}{2}mL)$, and its square, in the formulas (103)–(106), are to be interpreted when $mL \gg 1$. This factor is the evaluation at threshold of the factor $\sin^2(\frac{1}{2}L(\omega - \sqrt{\omega^2 - m^2})) \simeq \sin^2(\frac{1}{2}L(m - \sqrt{2m(\omega - m)}))$. When $\frac{1}{2}L\sqrt{2m\Delta} \gg 1$, the sine function has many oscillations over the integration interval in (106), (104), and it then can be replaced by the respective averages $\langle \sin^2(\frac{1}{2}L(m - \sqrt{2m(\omega - m)})) \rangle_{AV} = 1/2$, and $\langle \sin^4(\frac{1}{2}L(m - \sqrt{2m(\omega - m)})) \rangle_{AV} = 3/8$, in (106) and (104) respectively. When $\frac{1}{2}L\sqrt{2m\Delta}$ is of order unity, the full ω -dependent form of the argument of the sine function should be included in the integrals in (106), (104) in place of its threshold evaluation, and the integrals can then be computed numerically.

APPENDIX A: MATRICES FOR MATCHING AT $z = 0$ AND $z = L$

The matrices M_1 , M_2 , and $M_{12} = M_1^{-1}M_2$ needed for the $z = 0$ match are

$$\begin{aligned}
 M_1 &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ k & -k & 0 & 0 \\ 0 & 0 & \omega & -\omega \end{pmatrix}, \\
 M_2 &= \begin{pmatrix} 1 & 1 & \delta & \delta \\ \delta & \delta & 1 & 1 \\ k_- & -k_- & \delta k_+ & -\delta k_+ \\ \delta k_- & -\delta k_- & k_+ & -k_+ \end{pmatrix}, \\
 M_{12} &= \frac{1}{2} \begin{pmatrix} 1+k_-/k & 1-k_-/k & \delta(1+k_+/k) & \delta(1-k_+/k) \\ 1-k_-/k & 1+k_-/k & \delta(1-k_+/k) & \delta(1+k_+/k) \\ \delta(1+k_-/\omega) & \delta(1-k_-/\omega) & 1+k_+/\omega & 1-k_+/\omega \\ \delta(1-k_-/\omega) & \delta(1+k_-/\omega) & 1-k_+/\omega & 1+k_+/\omega \end{pmatrix}. \tag{A1}
 \end{aligned}$$

The matrices M_3 , M_4 , and $M_{34} = M_3^{-1}M_4$ needed for the $z = L$ match are

$$\begin{aligned}
 M_3 &= \begin{pmatrix} e^{iLk_-} & e^{-iLk_-} & \delta e^{iLk_+} & \delta e^{-iLk_+} \\ \delta e^{iLk_-} & \delta e^{-iLk_-} & e^{iLk_+} & e^{-iLk_+} \\ k_- e^{iLk_-} & -k_- e^{-iLk_-} & \delta k_+ e^{iLk_+} & -\delta k_+ e^{-iLk_+} \\ \delta k_- e^{iLk_-} & -\delta k_- e^{-iLk_-} & k_+ e^{iLk_+} & -k_+ e^{-iLk_+} \end{pmatrix}, \\
 M_4 &= \begin{pmatrix} e^{iLk} & e^{-iLk} & 0 & 0 \\ 0 & 0 & e^{iL\omega} & e^{-iL\omega} \\ k e^{iLk} & -k e^{-iLk} & 0 & 0 \\ 0 & 0 & \omega e^{iL\omega} & -\omega e^{-iL\omega} \end{pmatrix}, \\
 M_{34} &= \frac{1}{2(1-\delta^2)} \\
 &\times \begin{pmatrix} (1+k/k_-)e^{iL(k-k_-)} & (1-k/k_-)e^{-iL(k+k_-)} & -\delta(1+\omega/k_-)e^{iL(\omega-k_-)} & -\delta(1-\omega/k_-)e^{-iL(\omega+k_-)} \\ (1-k/k_-)e^{iL(k+k_-)} & (1+k/k_-)e^{-iL(k-k_-)} & -\delta(1-\omega/k_-)e^{iL(\omega+k_-)} & -\delta(1+\omega/k_-)e^{-iL(\omega-k_-)} \\ -\delta(1+k/k_+)e^{iL(k-k_+)} & -\delta(1-k/k_+)e^{-iL(k+k_+)} & (1+\omega/k_+)e^{iL(\omega-k_+)} & (1-\omega/k_+)e^{-iL(\omega+k_+)} \\ -\delta(1-k/k_+)e^{iL(k+k_+)} & -\delta(1+k/k_+)e^{-iL(k-k_+)} & (1-\omega/k_+)e^{iL(\omega+k_+)} & (1+\omega/k_+)e^{-iL(\omega-k_+)} \end{pmatrix}. \tag{A2}
 \end{aligned}$$

APPENDIX B: MATRIX ELEMENTS OF S_{ij}

$$2(\delta^2 - 1) k_- k_+ k S_{11} = e^{iLk} \left[i k_+ (k_-^2 + k^2) \sin(Lk_-) - 2k_+ k_- k \cos(Lk_-) \right. \\ \left. - \delta^2 [i k_- (k_+^2 + k^2) \sin(Lk_+) - 2k_+ k_- k \cos(Lk_+)] \right] \quad (B1)$$

$$2(\delta^2 - 1) k_- k_+ k S_{12} = i e^{-iLk} \left[k_+ (k_-^2 - k^2) \sin(Lk_-) + \delta^2 k_- (k^2 - k_+^2) \sin(Lk_+) \right] \quad (B2)$$

$$2(\delta^2 - 1) k_- k_+ k S_{13} = \delta e^{iL\omega} \left[(\omega + k) k_- k_+ [\cos(Lk_-) - \cos(Lk_+)] \right. \\ \left. - i [k_+ (k_-^2 + \omega k) \sin(Lk_-) - k_- (k_+^2 + \omega k) \sin(Lk_+)] \right] \quad (B3)$$

$$2(\delta^2 - 1) k_- k_+ k S_{14} = \delta e^{-iL\omega} \left[(\omega - k) k_- k_+ [\cos(Lk_+) - \cos(Lk_-)] \right. \\ \left. - i [k_+ (k_-^2 - \omega k) \sin(Lk_-) - k_- (k_+^2 - \omega k) \sin(Lk_+)] \right] \quad (B4)$$

$$S_{21} = -e^{2iLk} S_{12} \quad (B5)$$

$$2(\delta^2 - 1) k_- k_+ k S_{22} = e^{-iLk} \left[-k_+ [2k_- k \cos(Lk_-) + i(k_-^2 + k^2) \sin(Lk_-)] \right. \\ \left. + \delta^2 k_- [2k_+ k \cos(Lk_+) + i(k_+^2 + k^2) \sin(Lk_+)] \right] \quad (B6)$$

$$2(\delta^2 - 1) k_- k_+ k S_{23} = -e^{iL\omega} \delta \left[k_+ [k_- (\omega - k) \cos(Lk_-) - i(k_-^2 - \omega k) \sin(Lk_-)] \right. \\ \left. - k_- [k_+ (\omega - k) \cos(Lk_+) - i(k_+^2 - \omega k) \sin(Lk_+)] \right] \quad (B7)$$

$$2(\delta^2 - 1) k_- k_+ k S_{24} = e^{-iL\omega} \delta \left[-k_- [k_+ (\omega + k) \cos(Lk_+) + i(k_+^2 + \omega k) \sin(Lk_+)] \right. \\ \left. + k_+ [k_- (\omega + k) \cos(Lk_-) + i(k_-^2 + \omega k) \sin(Lk_-)] \right] \quad (B8)$$

$$S_{31} = -e^{-iL(\omega-k)} \frac{k}{\omega} S_{13} \quad (B9)$$

$$S_{32} = e^{-iL(\omega+k)} \frac{k}{\omega} S_{23} \quad (B10)$$

$$2(\delta^2 - 1) \omega k_- k_+ S_{33} = i e^{iL\omega} \left[k_- [2i\omega k_+ \cos(Lk_+) + (\omega^2 + k_+^2) \sin(Lk_+)] \right. \\ \left. - \delta^2 k_+ [2i\omega k_- \cos(Lk_-) + (\omega^2 + k_-^2) \sin(Lk_-)] \right] \quad (B11)$$

$$2(\delta^2 - 1) \omega k_- k_+ S_{34} = i e^{-iL\omega} \left[k_- (k_+^2 - \omega^2) \sin(Lk_+) - \delta^2 k_+ (k_-^2 - \omega^2) \sin(Lk_-) \right] \quad (B12)$$

$$S_{41} = e^{iL(\omega+k)} \frac{k}{\omega} S_{14} \quad (B13)$$

$$S_{42} = -e^{iL(\omega-k)} \frac{k}{\omega} S_{24} \quad (B14)$$

$$S_{43} = -e^{2iL\omega} S_{34} \quad (B15)$$

$$2(\delta^2 - 1) \omega k_- k_+ S_{44} = e^{-iL\omega} \left[-k_- [2\omega k_+ \cos(Lk_+) + i(\omega^2 + k_+^2) \sin(Lk_+)] \right. \\ \left. + \delta^2 k_+ [2\omega k_- \cos(Lk_-) + i(\omega^2 + k_-^2) \sin(Lk_-)] \right] \quad (B16)$$

The matrix elements of S have a large number of symmetries that arise from the structure of M_{12} and M_{34} . First, all of the S_{ij} are even functions of k_+ and even functions of k_- . Second, the matrix elements above have one of the following two structures: (i) $[\delta/(1-\delta^2)][f(k_+, k_-, k, \omega) - f(k_-, k_+, k, \omega)]$, (ii) $[1/(1-\delta^2)][\delta^2 f(k_+, k_-, k, \omega) - f(k_-, k_+, k, \omega)]$, with the functions f different for each matrix element. Since we have seen that under the interchange $k_+ \leftrightarrow k_-$ one has $\delta \leftrightarrow 1/\delta$, these two structures are both invariant under the interchange of k_+ and k_- , and so all matrix elements S_{ij} are invariant under this interchange. Finally, there are a number of relations between matrix elements under reversal of the sign of k or ω , as follows

$$\begin{aligned} S_{11} = S_{22}|_{k \rightarrow -k} , \quad S_{21} = S_{12}|_{k \rightarrow -k} , \quad S_{32} = S_{31}|_{k \rightarrow -k} , \quad S_{13} = S_{23}|_{k \rightarrow -k} , \quad S_{41} = S_{42}|_{k \rightarrow -k} , \quad S_{14} = S_{24}|_{k \rightarrow -k} , \\ S_{33} = S_{44}|_{\omega \rightarrow -\omega} , \quad S_{34} = S_{43}|_{\omega \rightarrow -\omega} , \quad S_{13} = S_{14}|_{\omega \rightarrow -\omega} , \quad S_{23} = S_{24}|_{\omega \rightarrow -\omega} , \quad S_{41} = S_{31}|_{\omega \rightarrow -\omega} , \quad S_{42} = S_{32}|_{\omega \rightarrow -\omega} . \end{aligned} \quad (\text{B17})$$

These relations serve as useful checks on the calculation.

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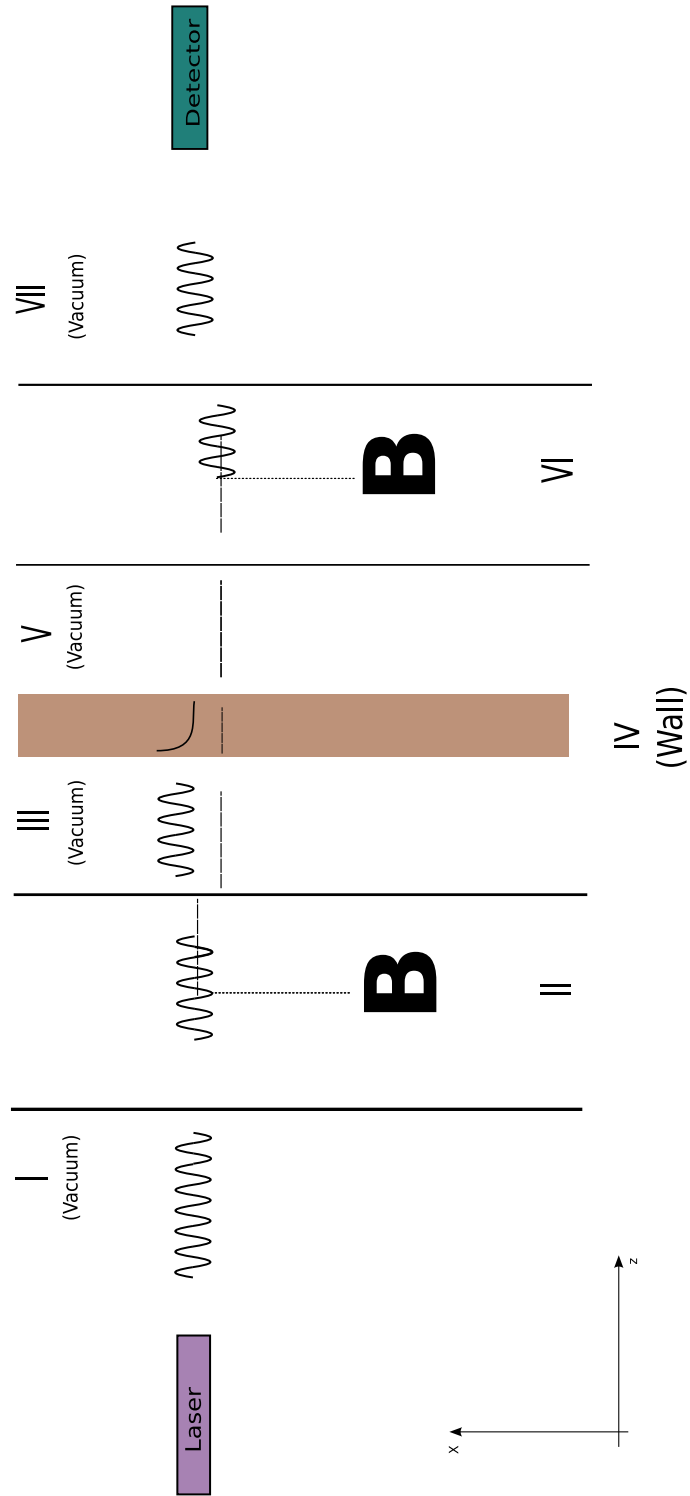


FIG. 1: Light shining through a wall setup. The wavy line indicates the photon field and the dashed line the axion field. In region I, there is only the photon wave, in region II the magnetic field gives rise to an axion wave, which exits with the photon wave into region III. In region IV, the “wall”, the photon wave is absorbed, leaving only the axion wave in region V. In region VI, the magnetic field regenerates a photon wave from the axion, which exits into region VII, where the regenerated photon is detected.